$$\vec{S}_r \times \vec{S}_\theta = \det\begin{bmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix} = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\cos^2\theta + r\sin^2\theta \rangle = r\langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle$$

$$\text{curl}(\vec F)(\vec S(r,\theta)) \cdot (\vec S_r \times \vec S_\theta)$$
$$= -r(2r^2\sin(\theta)\cos(\theta) + 2(1-r^2)r\sin(\theta) + r\cos(\theta))$$
$$= -r^2(r\sin(2\theta) + 2(1-r^2)\sin(\theta) + \cos(\theta))$$
$$\int_{\partial S} \vec F \cdot dr = \iint_S \text{curl}(\vec F)\cdot d\vec S$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} -r^2(r\sin(2\theta) + 2(1-r^2)\sin(\theta) + \cos(\theta))\, d\theta\, dr$$

$$= \int_{r=0}^{1}\left[-\tfrac{1}{2}r\cos(2\theta) - 2(1-r^2)\cos(\theta) + \sin(\theta)\right]\Big|_0^{\pi/2} dr$$

$$= \int_{r=0}^{1} -r^2\left[-\tfrac{1}{2}r(-1-1) - 2(1-r^2)(0-1) + (1-0)\right] dr$$

$$= \int_{r=0}^{1} -r^2(r + 2(1-r^2) + 1)\, dr = \int_{r=0}^{1} -r^2(-2r^2 + r + 3)\, dr$$

$$= \int_{r=0}^{1} 2r^4 - r^3 - 3r^2\, dr = \frac{2}{5}r^5 - \frac{1}{4}r^4 - r^3\Big|_{r=0}^{1} = \frac{2}{5} - \frac{1}{4} - 1 - 0 = \frac{2}{5} - \frac{1}{4} - 1$$

12/6/21 **Divergence Theorem:**

Idea: get another generalization of green's theorem

we saw, that we could state Green's Theorem as:

$$\int_{\partial D} F \cdot \hat n\, ds = \iint_D \text{div}(\vec F)\, dA$$

Divergence Theorem: Suppose that R is a simple solid region in $\mathbb{R}^3$ w/ a piecewise smooth bounding surface w/ one component →

Note: a simple solid is a region of $\mathbb{R}^3$ which "has no holes" and has one component to its bounding surface

← non example

i.e. the solid has a parametrization in any of the integration orders (i.e. $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$ etc.)

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Divergence theorem cont.

Thm $\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\, dV$

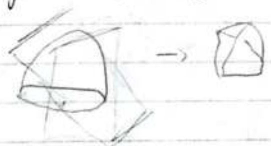
Ex: compute the Flux of $\vec{F} = \langle z, y, x \rangle$ across the unit sphere at the origin

Sol: we're asked to compute $\iint_S \vec{F}\cdot d\vec{S}$. we know $S = \partial R$ for R the solid unit disk at the origin.

$$\iint_S \vec{F}\cdot d\vec{S} \underset{S=\partial R}{=} \iint_{\partial R} \vec{F}\cdot d\vec{S} \underset{\text{divergent theorem}}{=} \iiint_R \overbrace{(0+1+0)}^{\operatorname{div}(\vec{F})}\, dV = \iiint_R 1\, dV = \operatorname{Vol}(R) = \frac{4}{3}\pi(1^3) = \frac{4}{3}\pi$$

Ex: compute $\iint_S \vec{F}\cdot d\vec{S}$ for $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ for S the surface of the region R bounded by $z = 1 - x^2 - y^2$, $z = 0$, $y = 0$, $y + z = 0$

Picture



$R = \{(x, y, z) : -y \le z \le 1 - x^2 - y^2,\ -\sqrt{1-x^2} \le y \le 0,\ -1 \le x \le 1$

Sol: Applying divergence theorem:

$\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[y^2 + e^{xz^2}] + \frac{\partial}{\partial z}[\sin(xy)] = y + 2y = 3y$

Now we can parametrize R in cylindrical coordinates via $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$

$R_{cy} = \{(r, \theta, z) : -r\sin(\theta) \le z \le 1 - r^2,\ 0 \le r \le 1$

Mistake

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change bounds to $z=1-x^2-y^2$, $z=0$

$\iint_{\partial R}\vec{F}\cdot d\vec{S} = \iiint_R \text{div}(\vec{F})\,dv$

$\text{div}(\vec{F}) = \nabla\cdot\vec{F} = 3y$ $\quad R_{cyl}: \{(r,\theta,z): 0\le r\le 1, 0\le\theta\le 2\pi, 0\le z\le 1-r^2\}$

$= \iiint_{R_{cy}} \text{div}(\vec{F})(r,\theta,z)\, r\, dV_{cyl} = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{1-r^2} 3r^2\sin(\theta)\,dz\,d\theta\,dr$

$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 3r^2\sin(\theta)[z]\Big|_{z=0}^{1-r^2} d\theta\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 3r^2(1-r^2)\sin(\theta)\,d\theta\,dr$

$= \int_{r=0}^{1} 3r^2(1-r^2)[-\cos(\theta)]\Big|_{\theta=0}^{2\pi} dr = \int_{r=0}^{1} 0\,dr = 0$

Exercise: repeat for R bounded by $z=1-x^2-y^2$ $z=0$ w/ $y\le 0$

Ex: Calculate the Flux of $\vec{F} = \langle xe^y, z-e^y, -xy\rangle$ across the ellipsoid $x^2+2y^2+3z^2=4$

Sol: let's apply the divergence theorem:
R, the solid ellipsoid yields
$\iint_{\partial R}\vec{F}\cdot d\vec{S} = \iiint_R \text{div}(F)\,dv$

divide all by 6

R parametrized by a modification of spherical coordinates

$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{3})^2} + \frac{z^2}{(\sqrt{2})^2} = \frac{4}{6} = \frac{2}{3}$ $\quad \left(\frac{x}{\sqrt{6}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 + \left(\frac{z}{\sqrt{2}}\right)^2 = \frac{2}{3}$

$x = \sqrt{6}\rho\sin(\phi)\cos(\theta)$

$y = \sqrt{3}\rho\sin(\phi)\sin(\theta)$

$z = \sqrt{2}\rho\cos(\phi)$

under our substitution, $\left(\frac{x}{\sqrt{6}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 + \left(\frac{z}{\sqrt{2}}\right)^2 = \frac{2}{3}$ iff $\rho^2 = \frac{2}{3}$

therefore, to parametrize solid ellipsoid,

$R_{new} = \{(\rho,\theta,\phi): 0\le\theta\le 2\pi, 0\le\phi\le\pi, 0\le\rho\le\sqrt{\frac{2}{3}}$

→

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$\iiint_{R_{new}} \text{div}(\vec{F})_{new} \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| dV_{new}$ Jacobian : $6\rho^2 \sin(\phi)$

$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = e^y - e^y + 0 = 0$

$\therefore \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R 0 \, dv = 0$